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Improvement of the Calculation of Scattering Amplitudes with External Fermions

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Abstract

In this paper, we present an improvement of a method for computing scattering amplitudes that include external (polarized) fermions with the following features: the formulas are quite general and work for different kinematic configurations and different external masses, they are explicitly covariant, they do not depend on a specific representation of the Dirac matrices and they have a meaningful limit when the masses tend to 0. The results presented make use of some well known formulas describing the density matrices in terms of projection operators within a more general formalism. Since our formulas intend to be as general as possible, we also take into account the possibility of a transverse polarization for massless fermions.

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The notion of spin is somewhat difficult to express in a general, covariant formalism which would have also a meaningful limit for $m \to 0$, since the Poincaré little groups are different for $p^2 > 0$ and $p^2 = 0$. Therefore, for the computation of Feynman amplitudes, we cannot obtain general formulas that are exactly continuous in the limit $m \to 0$. Furthermore, when fermions are involved, one has to deal with spinors and Dirac matrices, and no particular representation of these matrices has some physical meaning. But for practical calculations involving spinors, a lot of people still make reference to a specific representation of these matrices, which leads to analytical results that are not explicitly covariant. Therefore, for the explicit calculation of some elements of the $\mathcal{S}$ matrix which involve fermions, there could be an improvement if one can express them within a very general, covariant, representation independent formula, where the spin degrees of freedom clearly appear, and where the limit $m \to 0$ is meaningful and rather straightforward. There already exist in the literature some very useful formulas (see [1, 2, 3]), and the purpose of this paper is to generalize them, keeping in mind that we want to respect the criteria described above. For instance, the definition of the "conjugate" spinor $\bar{\psi}$ by $\psi^\dagger\gamma^0$ is purely conventional [4], assuming this way that we take a unitary representation of the Dirac matrices. This actually is not imposed by a physical principle but is rather a way to normalize the lagrangian. These matrices are even not supposed to be hermitian or anti-hermitian, and most of the calculations in this paper are completely representation independent, except for the sake of illustration.

It is not claimed however that the formulas presented in this paper lead to faster algorithms for the calculation of huge Feynman amplitudes, and we must recall the reader that for this purpose, there are some quite fast calculation techniques [5, 6] using spinor inner products. However, these algorithms are very fast only for massless fermions and introducing the case of massive fermions requires much more complex calculations. It is possible that in some cases, the formulas presented in this paper could be competitive for writing Monte-Carlo programs. Our purpose here is mostly to give simple tools for calculating analytically some amplitudes that are not very large, in such a way that one can possibly see the main physical features of the amplitude just by looking at its expression. For spinor inner products, one makes use of polar coordinates which need to set several geometrical conventions, yielding calculations that are not explicitly Lorentz covariant. This is why, although these tools exist and have proved to be efficient, we have found also useful to present how to calculate some amplitudes in a way which respect the symmetries (physical or not) of the problem and which includes as few conventions as possible.

The paper is organized as follows. In the first section, we review some generalities about the Dirac equation, focusing especially on the subtleties that are very scarcely found in elementary textbooks, and which we found useful to gather here. Then we show on a simple example of calculation involving neutrinos, how one can in general express an amplitude in a representation independent and explicitly covariant way. The next part is

devoted to the mathematical derivation of our general formulas. For this purpose, we will redemonstrate the well-known formula giving the density matrix of a pure spinor state $u\bar{u} = (\not{p} + m)(1 + \not{s}\gamma^5)/2$, using some representation independent calculations. We then show how to compute Feynman amplitudes or its square using these projection operator, focusing especially on the possible singularities appearing in the phase space which must be taken under consideration for an implementation in a Monte-Carlo program. The advantage of the demonstration we used is that it shows the uniqueness of the decomposition of the density matrix, it exhibits clearly the spin degrees of freedom, and especially the transverse degrees of freedom naturally emerge for massless fermions.

1 Generalities about the Dirac Equation

1.1 From Klein-Gordon to Dirac

In this section, we will review some basic things about the Dirac equation, and we will also focus on some points that are present in the literature, but unfortunately in very few papers.

First, one must recall that the Dirac equation is obtained by looking for a factorization of the Klein Gordon equation:

$$[\partial_\mu g^{\mu\nu} \partial_\nu + m^2]\Psi = 0 \tag{1.1}$$

provided we choose the $(+---)$ metric convention, and which can be factored into:

$$(i\not{\partial} + m)(i\not{\partial} - m)\Psi \tag{1.2}$$

where $\not{\partial} = \partial_\mu \gamma^\mu$, the γ^μ are the Dirac matrices (4×4 complex matrices which must obey the anticommutation rules $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$). Our choice of a specific sign convention for the metric is of importance here. If we choose the $(-+++)$ signature, we must then replace m by im or, to keep the mass term real, γ^μ by $i\gamma^\mu$, which is not algebraically equivalent to γ^μ. This could lead to some confusions when one reads the large literature about the differences in the structure of the two Clifford algebras corresponding to the two possible sign convention for the metric, $(1, 3)$ or $(3, 1)$ (see [7] and also some more mathematical references [8, 9, 10, 11]). However, the resulting physics fortunately does not depend on this metric convention, and to show this, we shall include the two possible sign conventions by rewriting the Klein-Gordon equation in the following way:

$$[\eta \partial_\mu g^{\mu\nu} \partial_\nu + m^2]\Psi = 0 \tag{1.3}$$

where $\eta = +1$ if the signature is $(1,3)$ and $\eta = -1$ if the signature is $(3,1)$. Thus the factorization now reads:

$$(i\omega\partial\!\!\!/ + m)(i\omega\partial\!\!\!/ - m)\Psi \tag{1.4}$$

where ω is one square root of η, i.e. up to a sign. Then, we can also choose between the two operators $(i\omega\partial\!\!\!/ + m)$ and $(i\omega\partial\!\!\!/ - m)$ to define the Dirac equation which gives another sign ambiguity. Therefore, to reabsorb these ambiguities, we define the matrices $\tilde{\gamma}^\mu = \omega\gamma^\mu$. We will consequently denote $\tilde{v} = v_\mu\tilde{\gamma}^\mu$ where v_μ is any 4-vector. The new anticommutation rules are therefore $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta g^{\mu\nu}$. That is to say, the $\tilde{\gamma}^\mu$ matrices are in a representation of the $C(1,3)$ Clifford algebra whatever the sign convention for the metric is. The study of $C(3,1)$, with a completely different complex structure becomes therefore irrelevant on the physical point of view. We can now set the Dirac equation to be $(i\tilde{\partial\!\!\!/} - m)\Psi = 0$, and since we have just shown that the only relevant signature is $(1,3)$, we will assume throughout this paper that this metric is chosen (or equivalently that $\gamma^\mu = \tilde{\gamma}^\mu$).

Since the γ^μ matrices (or $\tilde{\gamma}^\mu$) do not transform like a four-vector, there must be a specific transformation for Ψ to ensure the Lorentz covariance of this equation[(2)]:

$$\Psi(x) \xrightarrow{\Lambda} S_\Lambda\Psi(\Lambda^{(-1)}x) = \exp\left(i\frac{\omega_{\mu\nu}}{2}\sigma^{\mu\nu}\right)\Psi((\Lambda^{(-1)})^\mu{}_\nu x^\nu) \tag{1.5}$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ (or equivalently $\frac{i}{2}[\tilde{\gamma}^\mu, \tilde{\gamma}^\nu]$), and $\omega_{\mu,\nu}$ is an antisymmetric tensor, depending on the Lorentz transformation one performs[(3)]. Thanks to this transformation, the Dirac equation $((i\tilde{\partial\!\!\!/} - m)\Psi = 0)$ is Lorentz invariant.

Now one can question how arbitrary are the Dirac matrices. First, it is easy to check that if we have a set of 4 Dirac matrices γ^μ, and S is an invertible 4×4 matrix, then the new set $\gamma'^\mu = S\gamma^\mu S^{-1}$ still obey the anticommutation rules, and can be used as well as the first set of matrices, provided we operate the transformation $\Psi' = S\Psi$ on the wavefunction. The very interesting property is that the converse is true, and we have summarized in the appendix the old proof given in [4]. This property will be used in the following section, where we will define the notion of charge conjugation. For this purpose,

[(2)] We recall the reader that by Lorentz group is understood the subgroup of $O(1,3)$ which is connected to the identity. It is also called the orthochronous Lorentz group and denoted $L_+^\uparrow$. Spinors are in a representation of its covering group $SL(2,C)$ (see [12, 13]), but are not in general in a representation of the full Lorentz group, covered by what one call the $Pin(1,3)$ group [14] (and $Pin(3,1)$ for $O(3,1)$). The fact that one consider only $L_+^\uparrow$ comes from the possibility of parity or time reversal violations (see [12, 15, 16, 17, 18, 19] and related experiments [20, 21]), but some authors argue that, for systems that are symmetric under the full Lorentz group, one can in principle see experimentally some differences between $Pin(1,3)$ and $Pin(3,1)$ [14].

[(3)] S_Λ obeys $S_\Lambda^{(-1)}\gamma^\mu S_\Lambda = \Lambda^\mu{}_\nu\gamma^\nu$. Using the expansion $e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots$ and $[\sigma^{\mu,\nu}, \gamma^\rho] = 2i(g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu)$, one gets this relation: $\Lambda^\mu{}_\nu = \exp[-2(\omega)]^\mu{}_\nu$.

we will also need the general solution of the Dirac equation, expressed in momentum space:

$$\psi = \int \frac{d^3\vec{p}}{(2\pi)^3 2p^0} \left(a(p)u(p)e^{-ip\cdot x} + b^\dagger(p)v(p)e^{ip\cdot x} \right) \tag{1.6}$$

where we have included the creation and annihilation operators ($a(p)$ annihilates a fermion with momentum p and $b^\dagger$ creates an antifermion). u and v are the associated spinors that are respectively solutions of $(\not{p} - m)u(p) = 0$ and $(\not{p} + m)v(p) = 0$[4].

1.2 Spinors and Charge Conjugation

The Dirac equation being given, we must now construct the corresponding lagrangian. For this purpose, we have to face the definition of adjoint spinors. Indeed, when one calculate a specific Feynman amplitude, one commonly use the definition $\bar{\psi} = \psi^\dagger \gamma^0$ and the property $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$. One must be careful to the fact that the use of γ^0 here is only conventional, and has very little to do with physics.

The main point is to see that whenever $\{\gamma^\mu, \mu \in [\![0..3]\!]\}$ obey the anticommutation rules, then $\{\gamma^{\mu\dagger}, \mu \in [\![0..3]\!]\}$ verify also the same anticommutation relations. We have then demonstrated (see in appendix) that these two basis are related by an interior automorphism, i.e. there exists an invertible matrix H_+ such that:

$$\forall \mu \in [\![0..3]\!] \quad \gamma^{\mu\dagger} = H_+ \gamma^\mu H_+^{-1} \tag{1.7}$$

and also it implies (using the Schur lemma) that H_+ verify the property $H_+^\dagger = \mathbf{h}_+ H_+$ with $\mathbf{h}_+$ a complex number of modulus one. Then the adjoint spinor is defined as $\bar{\psi} = \psi^\dagger H_+$. The Dirac lagrangian can then be constructed as follows:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \tag{1.8}$$

For the sake of unitarity of the scattering matrix, we need the classical lagrangian to be real, which implies $\mathbf{h}_+ = 1$, that is to say H_+ is hermitian[5]. For the class of unitary representations, we can take $H_+ = \gamma^0$. However, unitary representations have no more physical relevance than any other representation, and therefore we shall not suppose in

[4] We can remark that we could have chosen to write the Dirac equation with a $+m$ instead of $-m$, which is equivalent to make the transformation $\psi \to \gamma^5 \psi$ on the spinor, and it is in fact just a change in the representation of the gamma matrices (because it is equivalent to make the transformation $\gamma^\mu \to -\gamma^\mu = \gamma^5 \gamma^\mu \gamma^5 = \gamma^5 \gamma^\mu (\gamma^5)^{(-1)}$.

[5] When one changes the representation of the Dirac matrices through $\gamma^\mu = S\gamma'^\mu S^{-1}$, H_+ changes into $H'_+ = \lambda S^\dagger H_+ S$ like an hermitian form (its signature is $(2,2)$).

the following that H_+ is equal to γ^0. Yet, since H_+ is defined up to a complex constant, imposing the unitarity of H_+ is a good way to normalize the lagrangian. We will rather impose a normalization on the density matrices at the end of section 4.3. Although it leads to the same constraint at the end, it allows us to keep H_+ undetermined. In other words, we will normalize the density matrices instead of H_+.

Now, the notion of charge conjugation needs the introduction of the gauge field and we get $(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0$ for the equation of motion of the interacting fermion field. Taking now the complex conjugate we obtain the charge conjugated equation $(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi^c = 0$, provided we take the following definition for ψ^c:

$$\psi^c = C_-^{-1}\psi^* \tag{1.9}$$

$$\forall \mu \in [\![0..3]\!] \quad \gamma_\mu^* = -C_-\gamma_\mu C_-^{-1} \tag{1.10}$$

(Note that when one conjugates ψ, one shall not forget to take the hermitian conjugate of the creation and annihilation operators). This definition can be seen at the level of free spinors in the momentum representation: if we have $(p\!\!\!/ - m)u(p) = 0$, then after complex conjugation one gets $(p\!\!\!/ + m)C_-^{-1}u^*(p) = 0$. In other words, the charge conjugated spinor of a so-called **u** spinor becomes a **v** spinor, which is the intuitive definition of charge conjugation (by computing ψ^c one can make the following identifications: $u^c = C_-^{-1}v^*$ and $v^c = C_-^{-1}u^*$). The existence of C_- is actually given by the theorem we have proved in appendix relating sets of matrices that obey the anticommutation relations, which is the case of $-\gamma_\mu^*$[6]. C_- must satisfy $C_-^*C_- = \mathbf{c}_- I$ where $\mathbf{c}_-$ is a real constant[7].

Having defined charge conjugation, we can take the opportunity to define a Majorana spinor. A Majorana spinor is its own charge conjugated particle, which necessarily constrains this kind of particle to be neutral, and it must therefore obey the relation $\psi^c = \alpha\psi$ (α being a possible phase). It implies the following relations between creation-annihilation operators and **u** and **v** spinors:

[6] In the so-called Majorana representation of the Dirac matrices, C_- is just a multiple of the identity matrix, and if we look at another representation $\gamma'_\mu = S\gamma_\mu S^{-1}$ then the matrix C_- transforms with the formula $C'_- = \alpha S^* C_- S^{-1}$, where α can be any complex number different from 0.

[7] In fact, one can scale the matrix C_- such that $\mathbf{c}_- = \pm 1$. The sign of $\mathbf{c}_-$ depends on the convention we choose for the metric, but not on the representation of the Dirac matrices. Actually, one has $\mathbf{c}_- = +1$ for the convention $(+---)$ (see [8]) and $\mathbf{c}_- = -1$ if one uses the $(-+++)$ convention, like in the book of Jauch and Rohrlich [4]. In this latter case, charge conjugation becomes an anti-involution, and we could wrongly state that Majorana spinors (spinors which are identical to their charge-conjugated version) do not exist in this case, and the physics would depend on the convention for the metric. The point is in this case that the rule given to define charge conjugation would change, and we should have with this convention $\psi^c = C_+^{-1}\psi$ where $\gamma_\mu^* = C_+\gamma_\mu C_+^{-1}$ and $C_+^*C_+ = \mathbf{c}_+ I$ ($\mathbf{c}_+ > 0$). One can obviously relate C_+ and C_- by $C_+ = C_-\gamma^5$.

$$b = a \tag{1.11}$$
$$C_-^{-1}\mathbf{v}^* = \alpha\mathbf{u} \tag{1.12}$$
$$C_-^{-1}\mathbf{u}^* = \alpha\mathbf{v} \tag{1.13}$$

And for the sake of consistency, one should have $C_-^* C_- = \frac{1}{|\alpha|^2} I$. This happens to be possible because the sign of $\mathbf{c}_-$ is independent of the representation of the Dirac matrices, and in the Majorana representation of the Dirac matrices[8], C_- is proportional to the identity (and thus C_- can always be written in the form $C_- = \sqrt{\mathbf{c}_-} S^* S^{-1}$ where S is any 4×4 invertible matrix). In fact, eq. 1.12 and 1.13 don't tell us much more about the structure of the wave function of a Majorana fermion, and Majorana spinors are nothing but standard spinors. The main difference comes from the quantization of the field, and especially from eq. 1.11 which allows to construct specific lagrangians that may distinguish between Majorana and Dirac spinors. We won't give more details about Majorana fermions here and refer the reader to the literature on this topic.

In a similar way, we could also have defined an invertible matrix T_+ or T_- for the transposition of the Dirac matrices (which also obey the anticommutation rules), but the three operations are related and since T_- can be obtained by combining H_+ and C_-, we will not have to use it. However, it can be useful to get a relation between C_- and H_+ using the fact that hermiticity and complex conjugation are two commuting operations. For this purpose we write:

$${}^t\gamma_\mu = (\gamma_\mu^\dagger)^* = -H_+^* C_- \gamma_\mu C_-^{-1} (H_+^*)^{-1} \quad (= -T_- \gamma_\mu T_-^{-1}) \tag{1.14}$$
$${}^t\gamma_\mu = (\gamma_\mu^*)^\dagger = -(C_-^{-1})^\dagger H_+ \gamma_\mu H_+^{-1} C_-^\dagger \tag{1.15}$$

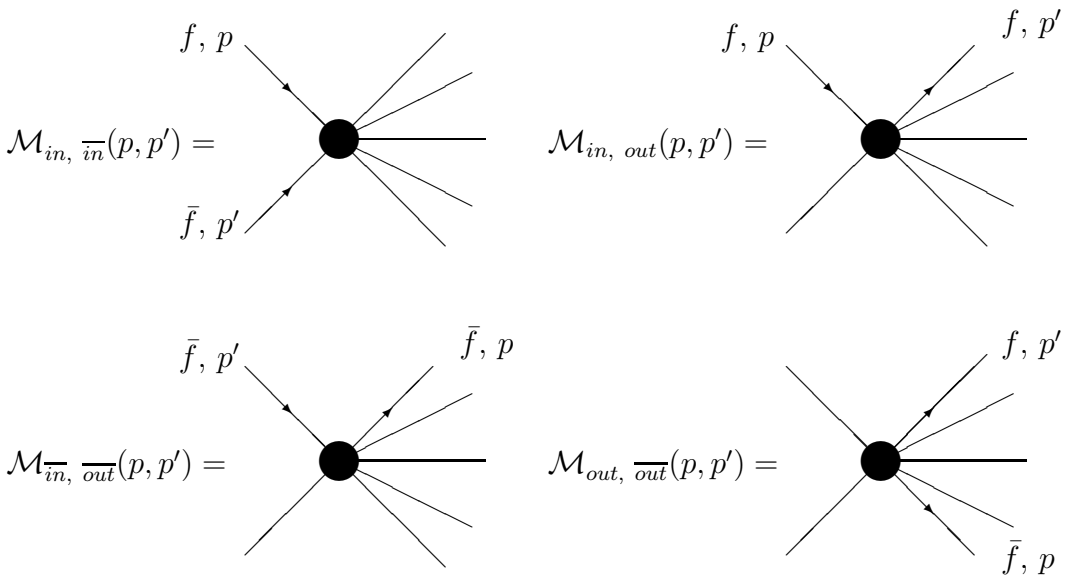
And using the Schur lemma, we know that there exists a non-vanishing constant κ such that:

$$H_+^* C_- = \kappa (C_-^{-1})^\dagger H_+ \tag{1.16}$$
$$H_+^{-1} C_-^\dagger = \kappa C_-^{-1} (H_+^*)^{-1} \tag{1.17}$$

Then, from a density matrix like $\rho = \psi\bar{\psi}$ (or more generally a rank 1 matrix like $\psi\bar{\psi}'$), we can compute its charge conjugated version $\psi^c\bar{\psi}^c$, and using the relations written in this section we may obtain easily:

[8] Note that the Majorana representation is only a kind of "eigenvector" for complex conjugation (up to a real inner automorphism), which is different from the notion of Majorana spinors. The existence of a Majorana representation (which depends only on the metric signature), only implies the possible existence of Majorana spinors.

Figure 1: *Amplitudes involving some fermion pairs*



$$\rho^c = \psi^c\bar{\psi}^c = C_-^{-1}\rho^*(H_+^*)^{-1}(C_-^{-1})^\dagger H_+ \quad (1.18)$$
$$= (\kappa)^{-1}C_-^{-1}\rho^* C_- \quad (1.19)$$

The last equation being obtained thanks to eq. 1.17, and if we require the lagrangian to be invariant under charge conjugation we must set $\kappa = 1$. We will see beyond (section 4.4) that charge conjugation, given by eq. 1.19 can be expressed as a condition on some Lorentz tensors which define completely the density matrix ρ.

2 General fermionic amplitudes

In this section, we sketch the way we intend to compute any scattering amplitude with at least one fermionic current. We shall first note that there are four different possible configurations corresponding to *a priori* four different expressions for the amplitudes (see fig. 1, where the overlined letters correspond to the antiparticles).

In a condensed notation, one may note a general amplitude as:

$$\mathcal{M} = \bar{\psi}' O \psi = tr[\mathcal{O}\psi\bar{\psi}'] = tr[\mathcal{O}\Sigma] \quad (2.20)$$

With $\Sigma = u(-cp)\bar{u}(-c'p')$ where $u(p)$ is a solution of $(\not{p} - m)u = 0$, and c, c' are the fermionic charge (*i.e.* -1 for a fermion, $+1$ for an antifermion), provided that p is the momentum corresponding to an **incoming** fermionic current, and p' is the momentum corresponding to an **outgoing** fermionic current (see fig 1). Of course, we could have chosen to write an amplitude by reference to an antifermionic current, or an annihilation or a pair production, but the four type of amplitudes are related and it is just a matter of convention. Also, we will parameterize the generic amplitudes using $u(-cp)$ for the two possible choices (fermions or antifermions) instead of using charge conjugation because we will see beyond that charge conjugation can be easily expressed in terms of changing the sign of the constant c or c' and also the sign of the spin degree of freedom which will be introduced in the next section. The general expression of the amplitude can be specialized to the four kinematic configurations of figure 1 in this way:

$$\begin{aligned}
\mathcal{M}_{in,\ \overline{in}} &= \bar{v}(p^i_+)\mathcal{O}u(p^i_-) = tr[\mathcal{O}u(p^i_-)\bar{u}(-p^i_+)] && (2.21)\\
\mathcal{M}_{in,\ out} &= \bar{u}(p^o_-)\mathcal{O}u(p^i_-) = tr[\mathcal{O}u(p^i_-)\bar{u}(p^o_-)] && (2.22)\\
\mathcal{M}_{\overline{in},\ \overline{out}} &= \bar{v}(p^o_+)\mathcal{O}v(p^i_+) = tr[\mathcal{O}u(-p^i_+)\bar{u}(-p^o_+)] && (2.23)\\
\mathcal{M}_{out,\ \overline{out}} &= \bar{u}(p^o_-)\mathcal{O}v(p^o_+) = tr[\mathcal{O}u(-p^o_+)\bar{u}(p^o_-)] && (2.24)
\end{aligned}$$

where p_- denotes the momentum of the fermion, p_+ is for the antifermion, and $\mathcal{O}$ is the interaction operator. Instead of decomposing spinors in their components, we will express the amplitudes without having to specify a specific representation for the Dirac matrices[9]. We shall therefore try to express the rank 1 matrix $\Sigma = u(-cp)\bar{u}(-c'p')$ in the basis of Dirac matrices, and we can do the same for the operator $\mathcal{O}$. Then the trace can be easily computed using simple algorithms[10]. If one prefer to compute directly the square of the amplitude, one has:

$$\begin{aligned}
|\mathcal{M}|^2 &= [\bar{u}(-c'p')\mathcal{O}u(-cp)]\,[\bar{u}(-c'p')\mathcal{O}u(-cp)]^\dagger\\
&= [\bar{u}(-c'p')\mathcal{O}u(-cp)]\left[\bar{u}(-cp)H_+^{-1}\mathcal{O}^\dagger H_+ u(-c'p')\right]\\
&= tr\left[\rho'\mathcal{O}\rho\bar{\mathcal{O}}\right] && (2.25)
\end{aligned}$$

with $\rho = u(-cp)\bar{u}(-cp)$, $\rho' = u(-c'p')\bar{u}(-c'p')$ and $\bar{\mathcal{O}} = H_+^{-1}\mathcal{O}^\dagger H_+$. If we decompose $\mathcal{O}$ on the basis of the Dirac matrices we have:

[9] Some variants of this formulation exist already in the literature, see [1, 2, 3]. Another calculation technique was developed by Hagiwara and Zeppenfeld [22], decomposing spinors in the chiral representation into their Weyl spinors. This method yields a quick algorithm for computation but here, we are looking for a formulation that is explicitly covariant and representation independent.

[10] Symbolic calculation programs can do this very easily nowadays. The fastest being probably FORM, and a rather convenient one is "M"[23].

$$\mathcal{O} = s_o + \bar{s}_o\gamma^5 + \not{v}_o + \not{a}_o\gamma^5 + S^o_{\mu\nu}\sigma^{\mu\nu} \tag{2.26}$$

and[11]:

$$\bar{\mathcal{O}} = s^*_o - \bar{s}^*_o\gamma^5 + \not{v}^*_o + \not{a}^*_o\gamma^5 + S^o_{\mu\nu}{}^*\sigma^{\mu\nu} \tag{2.27}$$

Then, in the following sections we will give the decomposition on the same basis for the density matrix ρ, which will give the final result for Σ and $\mathcal{M}$. Before, we shall discuss a simple example to show the practical procedure we will use in the general case.

3 A simple example

We will thus consider in this section the scattering of a neutrino, for instance in an electroweak process. The neutrino spinors can be easily expressed in terms of Weyl spinors if we take the chiral representation for the Dirac matrices. We get:

$$\Phi_L = \begin{pmatrix} -e^{-i\varphi/2}\sin(\theta/2) \\ e^{i\varphi/2}\cos(\theta/2) \end{pmatrix},\ \Phi_R = \begin{pmatrix} e^{-i\varphi/2}\cos(\theta/2) \\ e^{i\varphi/2}\sin(\theta/2) \end{pmatrix} \tag{3.28}$$

and using the notations $p_{(in,out)} = E_{(in,out)}(1, \vec{n}_{(in,out)})$, the Σ matrix can be computed with a little bit of algebra:

$$u(p_{in})\bar{u}(p_{out}) = \sqrt{E_{in}E_{out}} \begin{pmatrix} 0 & 0 \\ \Phi_L(p_{in})\Phi^\dagger_L(p_{out}) = \Omega & 0 \end{pmatrix} \tag{3.29}$$

$$= \sqrt{E_{in}E_{out}}\gamma^0 \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} = \ldots = \frac{1-\gamma^5}{2}\not{p} \tag{3.30}$$

One may then define the vector v^μ such that $v^0 = (\Omega_{11}+\Omega_{22})/2$, $v^1 = -(\Omega_{12}+\Omega_{21})/2$, $v^2 = -i(\Omega_{12}-\Omega_{21})/2$, $v^3 = -(\Omega_{11}-\Omega_{22})/2$. When the two fermions are not back-to-back, v^0 doesn't vanish and one can impose the condition that v^0 is real, multiplying by the suitable phase. One finally obtains the following form for v^μ:

$$v^\mu = \sqrt{E_{in}E_{out}} \begin{pmatrix} \sqrt{2}\sqrt{1+\vec{n}_{in}\cdot\vec{n}_{out}} \\ -\dfrac{\sqrt{2}}{\sqrt{1+\vec{n}_{in}\cdot\vec{n}_{out}}}(\vec{n}_{in}+\vec{n}_{out}+i\vec{n}_{in}\wedge\vec{n}_{out}) \end{pmatrix} \tag{3.31}$$

One can see that v^μ is orthogonal to both momenta of the fermions, and moreover, that the space-like part of this vector represents an elliptic polarization associated to the

[11]By $\not{v}^*$ we denote $v^*_\mu\gamma^\mu$. The Dirac matrices are not conjugated in this notation.

3-vector $\vec{n}_{in} - \vec{n}_{out}$. We have therefore combined the two spin (1/2) particles to obtain a spin-1 current.

But we will show that the expression of v^μ can be expressed in a covariant way through the formula:

$$v^\mu = \frac{(p_{out}\cdot k)p_{in}^\mu + (p_{in}\cdot k)p_{out}^\mu - (p_{out}\cdot p_{in})k^\mu - i\varepsilon^{\mu\nu\rho\sigma}k_\nu p_{in\,\rho}p_{out\,\sigma}}{\sqrt{2}\sqrt{2(p_{out}\cdot k)(p_{in}\cdot k) - (p_{out}\cdot p_{in})k^2}} \tag{3.32}$$

Where the four-vector k^μ serves as a reference. Expression 3.31 is actually obtained by setting $k^\mu = (1, \vec{0})$, but one can note that v^μ is invariant under the transformation $k \rightarrow k + \lambda\, p_{in} + \mu\, p_{out}$, which also implies that k must not lie in the plane generated by p_{in} and p_{out}. This will be rather obvious with the following. Note also that a rescaling of k does not affect v. Therefore, there is only one degree of freedom of k that may influence v, through a phase multiplication.

To obtain this expression, let's consider the density matrix for both the incoming and outgoing momenta (we will derive its expression in the following section):

$$u\bar{u} = \not{p}\frac{1+\gamma^5}{2} \tag{3.33}$$

then one has for a given vector k^μ:

$$u_{in}\bar{u}_{in}\not{k}u_{out}\bar{u}_{out} = (\bar{u}_{in}\not{k}u_{out})\Sigma = \not{p}_{in}\not{k}\not{p}_{out}\frac{1+\gamma^5}{2} = \not{p}\frac{1+\gamma^5}{2} \tag{3.34}$$

with:

$$v^\mu = (p_{out}\cdot k)p_{in}^\mu + (p_{in}\cdot k)p_{out}^\mu - (p_{out}\cdot p_{in})k^\mu - i\varepsilon^{\mu\nu\rho\sigma}k_\nu p_{in\,\rho}p_{out\,\sigma} \tag{3.35}$$

If k lies in the plane generated by p_{in} and p_{out}, v vanishes and the latter relations are useless. So let's take k outside this plane. Then we obtain the normalization factor in eq. 3.34 by multiplying once again by $\not{k}^{*(12)}$ and taking the trace, we get:

$$\begin{aligned} |\bar{u}_{in}\not{k}u_{out}|^2 &= (\bar{u}_{in}\not{k}u_{out})(\bar{u}_{out}\not{k}^* u_{in}) \\ &= tr\left[\not{p}\frac{1+\gamma^5}{2}\not{k}^*\right] = 2(v\cdot k^*) \\ &= 2\,(2\Re((p_{in}\cdot k)(p_{out}\cdot k^*)) - (p_{out}\cdot p_{in})(k\cdot k^*) \\ &\quad + i\varepsilon^{\mu\nu\rho\sigma}k_\mu k_\nu^* p_{in\,\rho}p_{out\,\sigma}) \end{aligned} \tag{3.36}$$

[12] $\not{k}^* \overset{def}{=} k^{*\mu}\gamma_\mu$ (see previous section).

$$\Rightarrow \bar{u}_{in}\not{k}u_{out} = e^{i\varphi}\sqrt{2}\sqrt{2\Re((p_{in}\cdot k)(p_{out}\cdot k^*)) - (p_{out}\cdot p_{in})(k\cdot k^*) + i\varepsilon^{\mu\nu\rho\sigma}k_\mu k^*_\nu p_{in\,\rho}p_{out\,\sigma}} \quad (3.37)$$

In this example, we have been obliged to introduce another 4-vector which has *a priori* no physical meaning (its role is only to set a reference for the phase of the amplitude), and we can be led into trouble if this 4-vector is not suitably chosen. This problem would not occur if we had computed directly the square of the amplitude, for which the introduction of this arbitrary 4-vector wouldn't be necessary. This also enlighten the impossibility to find a formalism for the amplitude which would be continuous in the massless limit, since this kind of singularity does not exist if one of the fermions is massive and k is chosen on the light cone. We will discuss this point further at the end of the paper, and we now turn to the general case of an amplitude that includes a fermionic current, represented by what we have called the Σ matrix.

4 The general case

4.1 General properties of the Σ matrix

In this section, we will specify the properties of the Σ matrix. Instead of expressing the spinors in a specific representation of the the Dirac matrices, we will express the general equations that determine Σ without having to take any specific representation. These equations are therefore more physically meaningful. This kind of formalism have already been studied a very long time ago by Pauli and others. It was called the bispinor algebra, and their intrinsic relations were studied in the context of Fierz identities (see [24]). We just give here another demonstration, which does not use Fierz identities which contain far too much information for our purpose. The main point of our proof being in fact a representation independent characterization of a rank-1 4×4 matrix. To proceed in the same way as in the previous example, we shall deal first with the density matrices ρ, for which we will use three constraints. The first is the equation of motion $(\not{p}-m)\rho = 0 = \rho(\not{p} - m)$. Then we will use a conjugation constraint $\bar{\rho} \stackrel{def}{=} H_+^{-1}\rho^\dagger H_+ = \rho$, and finally a criterion to say that ρ is a rank 1 matrix (using a characterization described in appendix). In principle, after these three steps, there should only remain a normalization condition to determine. So, let us first write the consequences of the equations of motion on ρ, provided we have decomposed ρ on a typical basis of Dirac matrices, that is:

$$\rho = s + \bar{s}\gamma^5 + \not{v} + \not{\bar{v}}\gamma^5 + S_{\mu\nu}\sigma^{\mu\nu} \quad (4.38)$$

where $S_{\mu\nu}$ is an antisymmetric tensor, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and one has the relations:

$$
\begin{aligned}
s &= \tfrac{1}{4}tr[\rho] \qquad \bar{s} = \tfrac{1}{4}tr[\rho\gamma^5] \\
v^\mu &= \tfrac{1}{4}tr[\rho\gamma^\mu] \quad a^\mu = \tfrac{1}{4}tr[\rho\gamma^5\gamma^\mu] \\
S_{\mu\nu} &= \tfrac{1}{8}tr[\rho\sigma^{\mu\nu}]
\end{aligned}
$$

Then the equations of motion can be also projected on the same basis and it gives [13]:

$$
\begin{aligned}
(\not{p} - m)\rho &= 0 \Leftrightarrow && (4.39)\\
p \cdot v &= ms && (4.40)\\
p \cdot a &= m\bar{s} && (4.41)\\
sp^\mu - 2iS^{\mu\nu}p_\nu &= mv^\mu && (4.42)\\
\bar{s}p^\mu - \varepsilon^{\alpha\beta\mu\nu}S_{\alpha\beta}p_\nu &= ma^\mu && (4.43)\\
\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}a_\rho p_\sigma - \frac{i}{2}[p,v]^{\mu\nu} &= mS^{\mu\nu} && (4.44)
\end{aligned}
$$

and:

$$
\begin{aligned}
\rho(\not{p} - m) &= 0 \Leftrightarrow && (4.45)\\
p \cdot v &= ms && (4.46)\\
p \cdot a &= -m\bar{s} && (4.47)\\
sp^\mu + 2iS^{\mu\nu}p_\nu &= mv^\mu && (4.48)\\
-\bar{s}p^\mu - \varepsilon^{\alpha\beta\mu\nu}S_{\alpha\beta}p_\nu &= ma^\mu && (4.49)\\
\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}a_\rho p_\sigma + \frac{i}{2}[p,v]^{\mu\nu} &= mS^{\mu\nu} && (4.50)
\end{aligned}
$$

These two sets of equations can be rearranged into a simpler one:

$$
\begin{aligned}
\bar{s} &= 0 && (4.51)\\
[p,v]^{\mu\nu} &= 0 \Leftrightarrow v^\mu = \lambda p^\mu && (4.52)\\
sp^\mu &= mv^\mu \Rightarrow s = \lambda m && (4.53)\\
p \cdot a &= 0 && (4.54)\\
p_\mu S^{\mu\nu} &= 0 && (4.55)\\
\varepsilon^{\mu\nu\alpha\beta}S_{\alpha\beta}p_\nu &= -ma^\mu && (4.56)\\
\varepsilon^{\mu\nu\alpha\beta}a_\alpha p_\beta &= 2mS^{\mu\nu} && (4.57)
\end{aligned}
$$

[13] $[p,v]^{\mu\nu} \stackrel{def}{=} p^\mu v^\nu - p^\nu v^\mu$

In the massive case, the two last equations are indeed equivalent. We can already see that the structure of the density matrix is quite constrained. Since $\bar{s} = 0$, the conjugation constraint simply says that all the remaining coefficients (s, v^μ, a^μ and $S_{\mu\nu}$) must be real (see eq. 2.27). We shall now use the characterization of ρ as a rank 1 matrix, which is the last constraint on ρ. In order to find the necessary and sufficient conditions for ρ to be a rank 1 matrix, we will apply the corollary of the second theorem demonstrated in appendix B, and in the first place we shall not make reference to the constraints we have already obtained with the equations of motion. We will add them at the end. The corollary tells us that we must have $tr(\rho Q \rho Q') = tr(\rho Q) tr(\rho Q')$ for every matrix Q and Q'. We will therefore choose for Q and Q' the different kind of Dirac matrices, which leads to[14]:

$\star$ For $Q = I$ we obtain respectively for $Q' = I,\ \gamma^5,\ \gamma^\mu,\ \gamma^\mu\gamma^5,\ \sigma^{\mu\nu}$:

$$
\begin{aligned}
2S_{\alpha\beta}S^{\alpha\beta} &= 3s^2 - \bar{s}^2 - (v^2 - a^2) && (4.58)\\
2s\bar{s} &= -i\varepsilon^{\mu\nu\rho\sigma} S_{\mu\nu} S_{\rho\sigma} && (4.59)\\
sv^\mu &= \varepsilon^{\lambda\alpha\beta\mu} a_\lambda S_{\alpha\beta} && (4.60)\\
sa^\mu &= \varepsilon^{\lambda\alpha\beta\mu} v_\lambda S_{\alpha\beta} && (4.61)\\
2sS^{\mu\nu} &= -\varepsilon^{\alpha\beta\mu\nu}(v_\alpha a_\beta + i\bar{s} S_{\alpha\beta}) && (4.62)
\end{aligned}
$$

Note that if $s \neq 0$, the lemma demonstrated in the appendix tells us that these 5 conditions 4.58 to 4.62 are sufficient to ensure that Σ is a rank 1 matrix. However, to treat also the case where $s = 0$, we must set Q to the other kind of "Dirac matrices".

$\star$ For $Q = \gamma^5$ we obtain ($Q' = \gamma^5,\ \gamma^\mu,\ \gamma^\mu\gamma^5,\ \sigma^{\mu\nu}$):

$$
\begin{aligned}
2S_{\alpha\beta}S^{\alpha\beta} &= 3\bar{s}^2 - s^2 + (v^2 - a^2) && (4.63)\\
\bar{s}v^\mu &= 2ia_\alpha S^{\alpha\mu} && (4.64)\\
\bar{s}a^\mu &= 2iv_\alpha S^{\alpha\mu} && (4.65)\\
2\bar{s}S^{\mu\nu} &= -is\varepsilon^{\mu\nu\rho\sigma} S_{\rho\sigma} + i(v^\mu a^\nu - v^\nu a^\mu) \\
&\quad + \frac{1}{4}(S^\mu{}_\alpha S_{\rho\sigma}\varepsilon^{\nu\alpha\rho\sigma} - S^\nu{}_\alpha S_{\rho\sigma}\varepsilon^{\mu\alpha\rho\sigma}) && (4.66)
\end{aligned}
$$

$\star$ For $Q = \gamma^\alpha$ and $Q' = \gamma^\beta,\ \gamma^\beta\gamma^5,\ \sigma^{\mu\nu}$ we get:

$$
\begin{aligned}
0 &= g^{\alpha\beta}(-a^2 - v^2 + s^2 - \bar{s}^2 + 2S_{\rho\sigma}S^{\rho\sigma}) - 2v^\alpha v^\beta + 2a^\alpha a^\beta + 8S^\alpha{}_\lambda S^{\lambda\beta} && (4.67)\\
0 &= g^{\alpha\beta}(a\cdot v) + (v^\alpha a^\beta - v^\beta a^\alpha) - s\varepsilon^{\alpha\beta\rho\sigma} S_{\rho\sigma} + 2i\bar{s}S^{\alpha\beta} && (4.68)\\
0 &= i\bar{s}(a^\mu g^{\alpha\nu} - a^\nu g^{\alpha\mu}) + sa_\lambda \varepsilon^{\lambda\alpha\mu\nu} \\
&\quad + 2v_\lambda(S^{\lambda\mu}g^{\alpha\nu} - S^{\lambda\nu}g^{\alpha\mu}) + 2(-v^\alpha S^{\mu\nu} + v^\mu S^{\alpha\nu} - v^\nu S^{\alpha\mu}) && (4.69)
\end{aligned}
$$

[14] We use here the convention $\varepsilon_{0123} = 1$.

$\star$ For $Q = \gamma^\alpha\gamma^5$ and $Q' = \gamma^\beta\gamma^5,\ \sigma^{\mu\nu}$, we obtain:

$$\begin{aligned} 0 &= g^{\alpha\beta}(-a^2 - v^2 - s^2 + \bar{s}^2 - 2S_{\rho\sigma}S^{\rho\sigma}) + 2v^\alpha v^\beta - 2a^\alpha a^\beta - 8S^\alpha{}_\lambda S^{\lambda\beta} \qquad (4.70)\\ 0 &= i\bar{s}(g^{\alpha\mu}v^\nu - g^{\alpha\nu}v^\mu) - sv_\lambda\varepsilon^{\lambda\alpha\mu\nu} \\ &\quad +2a_\lambda(g^{\alpha\mu}S^{\lambda\nu} - g^{\alpha\nu}S^{\lambda\mu}) + 2(a^\alpha S^{\mu\nu} - a^\mu S^{\alpha\nu} + a^\nu S^{\alpha\mu}) \qquad (4.71)\end{aligned}$$

$\star$ For $Q = \sigma^{\mu\nu}$ and $Q' = \sigma^{\rho\sigma}$, we obtain:

$$\begin{aligned} 0 &= (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})(s^2 + \bar{s}^2 + v^2 - a^2 + 2S_{\alpha\beta}S^{\alpha\beta}) \\ &\quad +2\left(g^{\mu\rho}(a^\nu a^\sigma - v^\nu v^\sigma) - g^{\mu\sigma}(a^\nu a^\rho - v^\nu v^\rho) - g^{\nu\rho}(a^\mu a^\sigma - v^\mu v^\sigma) + g^{\nu\sigma}(a^\mu a^\rho - v^\mu v^\rho)\right) \\ &\quad +8(g^{\nu\sigma}S^\mu{}_\lambda S^{\lambda\rho} - g^{\nu\rho}S^\mu{}_\lambda S^{\lambda\sigma} - g^{\mu\sigma}S^\nu{}_\lambda S^{\lambda\rho} + g^{\mu\rho}S^\nu{}_\lambda S^{\lambda\sigma}) \\ &\quad -8(S^{\mu\nu}S^{\rho\sigma} + S^{\mu\sigma}S^{\nu\rho} - S^{\mu\rho}S^{\nu\sigma}) - 2is\bar{s}\varepsilon^{\mu\nu\rho\sigma} \qquad (4.72)\end{aligned}$$

In eq. 4.69 we can insert eq. 4.65 and it simplifies into:

$$v^\alpha S^{\mu\nu} - v^\mu S^{\alpha\nu} + v^\nu S^{\alpha\mu} = \frac{s}{2}a_\lambda\varepsilon^{\lambda\alpha\mu\nu} \qquad (4.73)$$

And similarly, inserting eq. 4.64 into eq. 4.71 we get:

$$a^\alpha S^{\mu\nu} - a^\mu S^{\alpha\nu} + a^\nu S^{\alpha\mu} = \frac{s}{2}v_\lambda\varepsilon^{\lambda\alpha\mu\nu} \qquad (4.74)$$

Realizing the sum of eq. 4.67 and eq. 4.70 we obtain $a^2 = -v^2$, which then leads to $2S_{\alpha\beta}S^{\alpha\beta} = 2(s^2 + \bar{s}^2)$ and $v^2 = s^2 - \bar{s}^2$ using eq. 4.58 and eq. 4.63. The latter relations can be inserted in eq. 4.67 and eq. 4.70, the difference of which leads now to the simpler relation:

$$8S^\alpha{}_\lambda S^{\lambda\beta} = 2(v^\alpha v^\beta - a^\alpha a^\beta - s^2 g^{\alpha\beta}) \qquad (4.75)$$

Inserting this equation and the previous ones in eq. 4.72 we obtain:

$$S^{\mu\nu}S^{\rho\sigma} + S^{\mu\sigma}S^{\nu\rho} - S^{\mu\rho}S^{\nu\sigma} = -i\frac{s\bar{s}}{4}\varepsilon^{\mu\nu\rho\sigma} \qquad (4.76)$$

4.2 Solutions of the rank 1 equations

If we have the conditions $s \neq 0$ and $s^2 - \bar{s}^2 \neq 0$, the solutions of equations 4.58 to 4.62 are given by:

$$
\begin{aligned}
a \cdot v &= 0 && (4.77)\\
v^2 &= s^2 - \bar{s}^2 = -a^2 && (4.78)\\
S^{\mu\nu} &= \frac{-i}{2(s^2-\bar{s}^2)}\left[\bar{s}(v^\mu a^\nu - v^\nu a^\mu) - is\varepsilon^{\mu\nu\alpha\beta}v_\alpha a_\beta\right] && (4.79)
\end{aligned}
$$

The dual of $S^{\mu\nu}$ then reads:

$$
S^{*\mu\nu} = \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}S_{\rho\sigma} = \frac{-i}{2(s^2-\bar{s}^2)}\left[-s(v^\mu a^\nu - v^\nu a^\mu) + i\bar{s}\varepsilon^{\mu\nu\alpha\beta}v_\alpha a_\beta\right] \qquad (4.80)
$$

However, the relation 4.77 must always be true thanks to equation 4.68. The sum of eq. 4.67 and eq. 4.70 yields $a^2 = -v^2$ in any case, and the difference between eq. 4.63 and eq. 4.58 shows that the fundamental relation 4.78 must also be true in any case. Note also that in eq. 4.66, the last term could be surprising, but using 4.79, one finds that its contribution is 0. The remaining part of this equation is actually the "dual" equation of equation 4.62.

Now what happens if $\bar{s} = \pm s$. We can rewrite equation 4.62 using the new notations $S_\pm = S \pm S^*$ and $\omega_\pm = s \pm \bar{s}$:

$$
\omega_\pm S^{\mu\nu}_\pm = \pm\frac{1}{2}\left[i(v^\mu a^\nu - v^\nu a^\mu) \mp \varepsilon^{\mu\nu\rho\sigma}v_\rho a_\sigma\right] \qquad (4.81)
$$

Then if ω_+ or ω_- vanish, the corresponding self-dual tensor is left undetermined by this equation and a constraint appear between v and a.

4.3 Solution with the whole set of constraints

At this stage we can summarize all the constraints we have obtained. The equations of motion for the density matrix led us in the massive case to:

$$
\rho = \lambda(m + \not{p}) + \not{a}\gamma^5 + \frac{1}{2m}\varepsilon^{\mu\nu\alpha\beta}a_\alpha p_\beta \sigma_{\mu\nu} \qquad (4.82)
$$

with $a \cdot p = 0$ and the rank 1 characterization only adds the condition $a^2 = -\lambda^2 m^2$ because of course, a lot of the previous equations are redundant. The last two equations show that the 4-vectors $a/\lambda \pm p$ lie on the light cone. That is to say we can write $a = \lambda(p - k_1) = \lambda(k_2 - p)$ with $k_1^2 = k_2^2 = 0$, $k_1 \cdot p = k_2 \cdot p = m^2$. Actually, only one of these two vectors is useful since, if k_1 is given, one can take $2p - k_1$ for k_2. Therefore, let us consider any non-vanishing 4-vector k lying on the light cone. We can therefore set:

$$a^\mu = h\lambda\left(p^\mu - \frac{m^2}{p\cdot k}k^\mu\right) \tag{4.83}$$

Where $h = \pm 1$. The scale of k is of no importance in this expression, whereas its direction becomes a good candidate for defining the direction of reference for the polarization of the fermion. We can also look at the light-cone limit of this expression. If the space-like projection of p^μ does not tend to the one of k^μ, the limit is obviously the limit of $h\lambda p^\mu$, and the possible singularity occurs when p becomes almost collinear to k. In order to study this limit, we can set $p^\mu = k^\mu + \epsilon\delta^\mu$ with $\epsilon \to 0$ (we must have chosen k suitably here, and especially $k^0 > 0$). The expression of a^μ then reads:

$$\frac{a^\mu}{h\lambda} = -k^\mu\left(1 + \epsilon\frac{\delta^2}{\delta\cdot k}\right) + \epsilon\delta^\mu \tag{4.84}$$

And the limit for a^μ exists ($a^\mu \simeq -h\lambda p$). Rewriting the whole density matrix we get:

$$\begin{aligned}
\rho &= \lambda\left(m + \not{p} + h\left(\not{p} - \frac{m^2}{p\cdot k}\not{k}\right)\gamma^5 + \frac{mh}{2(p\cdot k)}\varepsilon^{\mu\nu\alpha\beta}p_\alpha k_\beta \sigma_{\mu\nu}\right) && (4.85)\\
&= \lambda\left(m + \not{p} + h\left(\not{p} - \frac{m^2}{p\cdot k}\not{k}\right)\gamma^5 + mh\left(1 - \frac{\not{p}\not{k}}{p\cdot k}\right)\gamma^5\right) && (4.86)\\
&= \lambda(\not{p} + m)\left(1 + h\left(1 - \frac{m\not{k}}{p\cdot k}\right)\gamma^5\right) && (4.87)
\end{aligned}$$

We recover for the massive case the well known result ($\rho = (\not{p}+m)\frac{1+\not{a}\gamma^5}{2}$ with $a\cdot p = 0$ and $a^2 = -1$), in a slightly different form. This last formula is already very often used in the literature, and one may question about the interest in redemonstrating it. The first interest is to use this formula (especially in the form of eq. 4.85) to compute scattering amplitudes or the modulus square of these amplitudes with external fermions, (as we will see in the following) in a way that respects the criteria we have defined in introduction. And also, it shows naturally how the spin degrees of freedom emerge and the possibility of some transverse degrees of freedom in the massless limit. In fact, when one looks at the limit $m \to 0$ in eq. 4.85, the last term vanish[(15)], and if $p\cdot k$ doesn't tend to 0, the term containing γ^5 becomes equivalent to $\not{p}\gamma^5$, whereas when $p\cdot k$ also tends to 0 when m vanish (i.e. $p^\mu \to Ck^\mu$), then the γ^5 term is equivalent to $-\not{p}\gamma^5$. The $m \to 0$ limit allows us to recover the two possible density matrices used in the literature for massless fermions ($\rho = \not{p}(1 \pm \gamma^5)/2$). However, in the massive case we had more freedom for this density matrix because the direction of k is free. In some sense, this freedom seems "frozen" in

[(15)]This limit is obvious if $p\cdot k$ does not tend to 0. Otherwise, if $p\cdot k \to 0$ when $m \to 0$, one has $p^\mu \to Ck^\mu$. Then one can see that the limit is 0 by taking for instance $p^\mu = (E = \sqrt{m^2+p^2}, 0, 0, p)$ and $k = (1, 0, 0, 1)$. Then $\varepsilon^{1203}(p_0k_3 - p_3k_0) = E - p = p\cdot k$ and the limit is clearly 0 also in this case.

the limit $m \to 0$, but it is not really the case, and we shall study in details the solutions of the constraint equations for $m = 0$ exactly. We can actually deduce from eq. 4.56 that the generic solution for $S^{\mu\nu}$ when $m = 0$ is $S^{\mu\nu} = \lambda[p, \varepsilon]^{\mu\nu}/2$ with $p \cdot \varepsilon = 0$ (eq. 4.55). Therefore, $S^{\mu\nu}$ is not necessarily 0, contrary to what we have deduced from the limit $m \to 0$. The rank 1 conditions then add another constraint, but relax also one. Since $a^2 = 0$ in our case, the normalization of a^μ is no more imposed (the coefficient λ in factor of eq. 4.83 is no more valid) and from eq. 4.57 one has only $a^\mu = \lambda' p^\mu$ with *a priori* $\lambda' \neq \lambda$. From eq. 4.75 we get a constraint on the normalization of ε^μ:

$$\lambda^2(1 + \varepsilon^2) = \lambda'^2 \tag{4.88}$$

We can therefore write the density matrix in the massless limit in the form:

$$\begin{aligned} \rho &= \not{p}\left(\lambda + \lambda'\gamma^5 + i\not{\varepsilon}\right) && (4.89)\\ &= \lambda\not{p}\left(1 + h\sqrt{1+\varepsilon^2}\gamma^5 + i\not{\varepsilon}\right) && (4.90) \end{aligned}$$

The emergence of the "transverse" degrees of freedom in the massless case is not really a surprise. Their origin comes from the fact that the little group of a massless momentum ($ISO(2)$) is of the same dimension (3) as the one of a massive momentum ($SO(3)$), and we should therefore obtain the same number of degrees of freedom. However, the transverse degrees of freedom of massless particles are not observed in experiments, and we are led to discard them for neutrinos (see [25] eq. 2.5.38). In the context of the study of some new physics where some massless fermions not yet observed could exist, there is *a priori* no reason to discard them and it is also a reason why we present the most general situation in this paper. Finally, for the global normalization of these density matrices, we choose $\lambda = 1/2$ to conform to a common normalization used in the literature.

4.4 Charge conjugation in this formulation

Since we have obtained the decomposition of the density matrices on the basis of Dirac matrices, we shall now study how charge conjugation operates on these decompositions. From what we have seen in section 1.2 we can write:

$$\begin{aligned} \rho^c &= C_-^{-1}\rho^* C_- && (4.91)\\ &= s^* I - \bar{s}^*\gamma^5 - \not{v}^* + \not{a}^*\gamma^5 - S^*_{\mu\nu}\sigma^{\mu\nu} && (4.92)\\ &= sI - \not{v} + \not{a}\gamma^5 - S_{\mu\nu}\sigma^{\mu\nu} && (4.93) \end{aligned}$$

The last equation comes from the hermiticity of ρ and the fact that we have $\bar{s} = 0$ in any case. Applying this on the explicit expression of ρ (eq. 4.85), the charge conjugation acts rather simply on ρ:

$$\rho(p, k, h) \to \rho^c(p, k, h) = \rho(p' = -p, k' = k, h' = -h) \tag{4.94}$$

or also this way:

$$\rho(p, k, h) \to \rho^c(p, k, h) = \rho\left(p' = -p, k' = 2p - \frac{m^2}{p \cdot k}k, h' = h\right) \tag{4.95}$$

We can see that it is equivalent to change the sign of h, keeping the vector of reference constant, or to keep h the same and to enforce a symmetry operation on k. Also eq. 4.94 has the advantage to apply in the massless case where we can also write from eq. 4.90:

$$\rho(p, \varepsilon, h) \to \rho^c(p, \varepsilon, h) = \rho(-p, \varepsilon, -h) \tag{4.96}$$

4.5 Computation of a generic amplitude

We have now all the elements necessary to compute a generic amplitude of the form $\mathcal{M} = tr[\mathcal{O}\Sigma]$. We therefore wish to proceed in a similar way as we did in the simple example of section 3. Thus we can write the equivalence:

$$u\bar{u}' \equiv \frac{u\bar{u}\not{k}u'\bar{u}'}{\sqrt{tr[u\bar{u}\not{k}u'\bar{u}'\not{k}]}} \tag{4.97}$$

where the 4-vector k serves as a reference for the phase of the amplitude and the equivalence means "up to a phase". It is important to note at this stage that we can be led into trouble if an amplitude contains several identical particles. In this case we should have the same reference for the phase of two diagrams where two identical fermions are permuted, because of the possible interferences. In this case, this method shouldn't work because we have arbitrarily changed the relative phase between the diagrams. In this case, a covariant expression can be obtained through the computation of the square of the amplitude, which often means huge analytical expressions. We shall not discuss further this problem. Coming back to our single fermionic current, we can generically write the Σ matrix in the form:

$$\Sigma = \frac{\rho_u \not{k} \rho_{u'}}{\sqrt{tr[\rho_u \not{k} \rho_{u'} \not{k}]}} \tag{4.98}$$

where one can replace ρ_u and $\rho_{u'}$ by their expression given in formula 4.85 for massive fermions or eq. 4.90 for massless fermions. Then, using eq 4.98, the amplitude can be written:

$$\mathcal{M} = \frac{tr[\mathcal{O}\rho_u \not{k} \rho_{u'}]}{\sqrt{tr[\rho_u \not{k} \rho_{u'} \not{k}]}} \tag{4.99}$$

and one has to compute two simpler traces, instead of one possibly huge trace if one wants to compute the square of the amplitude using eq. 2.25. As for the choice of k^μ, we have seen in the simple example presented in section 3 that we may have some problems if this vector is not suitably chosen, especially in the massless case, because of a kinematic singularity appearing in the plane generated by the two momenta. And if one intends to implement this formula inside a Monte-Carlo program, where the events are generated randomly, it is better to avoid this kind of singularity to be sure that none of the momentum configurations will be close to the singularity. Therefore in the massless case, one may choose k^μ such that $k^2 = 1$, and in the massive case, the explicit computation of the normalization factor in eq. 4.99 shows that one may preferably choose k^μ on the light cone. Since the explicit calculation of these traces can be done easily by some symbolic calculation programs, we shall let the reader do them if he is interested in such calculations, because it is often important to take advantage of the particular situations to choose k^μ, or the spin axes properly.

We may end this section by mentioning that we can also choose a simpler operator than $\not{k}$ to be inserted into the normalization factor when at least one of the fermions is massive. We can simply use the identity matrix instead of $\not{k}$ and we get:

$$\mathcal{M} = \frac{tr[\mathcal{O}\rho_u \rho_{u'}]}{\sqrt{tr[\rho_u \rho_{u'}]}} \tag{4.100}$$

Which is simpler to compute, and shall not lead to kinematical singularities in most cases. Since the denominator can be of order mm', it will be numerically better to use it for heavy fermions.

5 Conclusion

In this paper, we have shown a quite general method for the calculation of Feynman amplitudes or its square with external fermions. The mass of the fermions can be of any value, even in the same fermionic line. The formulas given are also independent of the representation of the Dirac matrices, explicitly covariant and have a meaningful massless limit. The formalism is not exactly continuous in $m = 0$ in the strict sense since

there are some degrees of freedom that appear in the case $m = 0$ that compensate the transverse degrees that are "frozen" into the two helicity modes when $m \to 0$. However, these transverse degrees of freedom are not observed for neutrinos and they are therefore not taken under consideration for these particles. For the study of some New Physics, it can be interesting to keep them. The other trouble with massless fermions comes when one computes a scattering amplitude with a current composed of two of these massless fermions. There can be some singularity in phase space for the normalization condition if one takes a fixed momentum for the reference phase, which can lay in the plane of the two external momenta for some kinematic configurations. This singularity does not exist for the computation of the square of the amplitude, which fortunately in the massless case may lead to expressions of reasonable size, contrary to the massive case.

To obtain the general expression of an amplitude, we started from well known formulas giving the density matrices that we have redemonstrated in the most general case. Using this kind of demonstration, the possibility of a transverse polarization for massless fermions naturally emerges, and we have also characterized within this formalism the notion of charge conjugation and Majorana spinors. We expect that these results can be useful to get simpler analytic expressions for short Feynman amplitudes, or not-so-short amplitudes computed using symbolic calculation programs. We also expect a simpler implementation of fermions in Monte-Carlo programs, thanks to the fact that the formalism presented in this paper requires very few conventions, which was also one of the goals of this work.

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Appendix A: some properties of the Dirac matrices

Suppose that we have one set of γ^μ matrices that obey the anticommutation rules. From this set of 4 matrices, one can explicit a basis of $\mathcal{M}_4(C)$[16], γ^r ($r \in [\![1..16]\!]$) in this way:

$$\gamma_r = {\gamma_0}^{\alpha_0(r)} {\gamma_1}^{\alpha_1(r)} {\gamma_2}^{\alpha_2(r)} {\gamma_3}^{\alpha_3(r)} \tag{5.101}$$

with $\alpha_\mu(r) = 0, 1$. We may also sometimes denote $n_\gamma(r) = \alpha_0(r)+\alpha_1(r)+\alpha_2(r)+\alpha_3(r)$. When $n_\gamma(r)$ is even, the matrix is said to be in the even subalgebra of the Clifford algebra.

[16] $\mathcal{M}_4(C)$ denotes the set of 4×4 complex matrices throughout the paper.

This basis is chosen essentially for the proof of the theorem relating two different sets of Dirac matrices (see beyond), but most of the time one uses the more convenient basis ($I = \gamma_{(r=1)}$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma_{(r=16)}$, γ^μ, $\gamma^\mu\gamma^5$, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$)[17].

With these definitions one can prove that $tr[\gamma_r] = 0$ if γ_r is not the identity matrix ($r \neq 1$) and that $\gamma_p\gamma_q = \eta_{p,q}\gamma_{I(p,q)}$, where $\eta_{p,q} = \pm 1$, and more importantly, for a fixed q, $p \mapsto I(p,q)$ is a permutation of $[\![1..16]\!]$ (and similarly for a fixed p, $q \mapsto I(p,q)$ is also a permutation). Also $\eta_{p,q}$ depends only on the anticommutation relations and the metric convention, and not on the specific set (representation) of Dirac matrices chosen. Since we have also shown in the beginning of the paper that the only relevant signature is $(1,3)$, we also assume this metric to be chosen. We have also $tr[\gamma_i\gamma_j] = C_{ij}$, where the coefficients $C_{i,j}$ vanish if and only if $i \neq j$.

Demonstration of the relation between different representations

Suppose now that we have two sets of Dirac matrices γ^μ and γ'^μ that obey the anticommutation rules. As before, we can define a basis for the 4×4 complex matrices γ_r, and another similar basis ${\gamma'}_r$ defined in the same way but using γ'^μ instead of γ^μ. We shall also need another 4×4 complex matrix, F upon which we do not impose any constraint at this time. Then we define the matrix S as:

$$S = \sum_{r=1..16} {\gamma'}_r F {\gamma_r}^{(-1)} \tag{5.102}$$

We will show that S must be 0 or invertible, and then that in this latter case one has $\gamma'^r = S\gamma^r S^{-1}$ (it is sufficient to check this property for $r \equiv \mu \in [\![0..3]\!]$).

The proof comes from the following relation:

$$\begin{aligned}
S\gamma_t &= \sum_{r=1..16} {\gamma'}_r F {\gamma_r}^{(-1)} \gamma_t \\
&= \sum_{s=1..16} {\gamma'}_{I(t,s)} F {\gamma_{I(t,s)}}^{(-1)} \gamma_t \qquad (\Leftarrow \quad r \to I(t,s)) \\
&= \sum_{s=1..16} {\gamma'}_{I(t,s)} F {\gamma_s}^{(-1)} {\gamma_t}^{(-1)} \gamma_t \eta_{t,s} \qquad (\Leftarrow \quad def.\ \ of\ \gamma_{I(t,s)}) \\
&= \sum_{s=1..16} \eta_{t,s} {\gamma'}_{I(t,s)} F {\gamma_s}^{(-1)} \\
&= \sum_{s=1..16} {\gamma'}_t {\gamma'}_s F {\gamma_s}^{(-1)} \qquad (\Leftarrow \quad def.\ \ of\ \gamma_{I(t,s)}) \\
&= {\gamma'}_t S
\end{aligned} \tag{5.103}$$

[17] Sometimes we will have to consider $\gamma_{(r=1)}$ which is just the identity, different from the usual $\gamma_{(\mu=1)}$. We will therefore write the "$r = ...$" when necessary, in order to avoid some confusions about the meaning of the indices.

This is the very fundamental property we need to prove our latest assertion. Now if x is a four vector such that $Sx = 0$, then every $\gamma_r x$ is in the kernel of S. It means that $Ker\, S$ is stable through the action of the whole algebra, that is to say every 4×4 complex matrix ($\mathcal{M}_4(C)$). If we consider an irreducible representation of the Dirac matrices, it implies that $Ker\, S$ must be an improper subspace, i.e. 0 or the whole spinor space C^4. Therefore, S is invertible or 0. Now, why is the four dimensional representation irreducible? Suppose that $K = Ker\, S$ is not 0, and let L be a subspace of C^4 such that $C^4 = K \oplus L$. Let $\mathcal{M}$ be an endomorphism of C^4 such that the image of one (non-zero) vector in K lies in L (it only needs L to be different from 0). Now, the key point is that $\mathcal{M}$ can be decomposed on the γ_r basis. Since K is stable by all the γ_r, we get a contradiction and L must be 0, $\Rightarrow K = C^4 \Rightarrow S = 0$. □

We have now demonstrated that two sets of Dirac matrices are related by an inner automorphism if we can find $S \neq 0$. And it is possible to find F such that S is not 0. Otherwise, for all $x, y \in C^4$, posing $F = yx^\dagger$, one has $\sum_r (y^\dagger {\gamma'}_r y) x^\dagger {\gamma_r}^{(-1)} = 0$, thus $\sum_r (y^\dagger {\gamma'}_r y) {\gamma_r}^{(-1)} = 0$ and taking the trace, one obtains $y^\dagger {\gamma'}_{(r=1)} y = 4 y^\dagger y$ which cannot be 0 if $F \neq 0$. So one can find F such that S is non-zero, then S is invertible and the two representations are related through an inner automorphism. □

Unitary representations

In this paragraph, we will show that one can construct a unitary representation of the Dirac matrices from any irreducible representation.

So, if we denote $H = \sum_r \gamma_r^\dagger \gamma_r$, one can use some arguments similar to the ones used in eq. 5.103 (with also the fact that $\eta_{p,q}^2 = 1$) to show that $\gamma_\mu^\dagger H \gamma_\mu = H$. Now, it is clear that H is hermitian definite and positive, which allows us to write it as a "square", i.e. $H = h^\dagger h$ and h is invertible[18]. Thus we have $\gamma_\mu^\dagger h^\dagger h \gamma_\mu = h^\dagger h$ and therefore the representation given by $h \gamma_\mu h^{-1}$ is unitary. □

Appendix B: mathematical proof of some elementary theorems

In this appendix we will demonstrate two interesting properties. The second one is used in the body of this paper. The first one is rather trivial if one uses a specific representation for the Dirac matrices. The aim here is just to illustrate that most of the important properties within the Dirac formalism can be shown without the need of any specific representation for the γ matrices.

[18] h is also unique if it is taken hermitian positive definite.

Theorem 1: *Let p be a non vanishing four momenta. We denote $m = \sqrt{p^2}$ if $p^2 \geq 0$ and $m = i\sqrt{-p^2}$ if $p^2 < 0$. Let c be a constant which can be set to ± 1 (if the convention for the metric is $(+---)$), or $\pm i$ (if the convention for the metric is $(-+++)$, case which can be avoided if we replace $\not{p}$ by $\tilde{p}$, as shown in the beginning of the paper). Then $Ker(\not{p} - cmI) = Im(\not{p} + cmI)$.*

Proof: Let M_4 be the four dimensional Minkowski space, and C^4 a four dimensional complex vector space, in which we define spinors.
Since $\not{p}^2 = c^2m^2I$, it is clear that $Im(\not{p} + cmI) \subset Ker(\not{p} - cmI)$. In the case $m \neq 0$, if $u \in Ker(\not{p} - cmI)$, then $u = (\not{p} + cmI)(u/(2cm))$, and the theorem is proved. In fact, the eigenvalues of $\not{p}$ are of order two, because $\not{p}$ has at most 2 non-vanishing eigenvalues, and its trace must be zero. Thus $\not{p} - cmI$ are two rank two matrices, and we recover the elementary result that the space of states for a spin 1/2 particle is of dimension 2. This is also true when $m = 0$ as we shall see.

Now suppose that $m = 0$. We know that $dim\ Ker(\not{p}) + dim\ Im(\not{p}) = 4$, and we have seen that $Im(\not{p}) \subset Ker(\not{p})$, and since p^μ is a non vanishing vector, $\not{p} \neq 0$ and one has: $1 \leq dim\ Im(\not{p}) \leq dim\ Ker(\not{p})$. Thus we have two possibilities:

- $dim\ Im(\not{p}) = 2 = dim\ Ker(\not{p})$. Then the theorem is proved.
- $dim\ Im(\not{p}) = 1$ and $dim\ Ker(\not{p}) = 3$.

We will therefore show that the second case is not possible. Let u_i, $i \in \{0, 1, 2, 3\}$ be a basis of C_4, such that $\{u_1, u_2, u_3\}$ is a basis of $Ker(\not{p})$ and $u_1 = \not{p}u_0$. The vector p^μ is of the form $\mathbf{p}(1, \vec{n})$ (with $\vec{n}^2 = 1$). Let $\vec{n}'$ and $\vec{n}''$ two vectors such that $\vec{n}'$, $\vec{n}''$ and $\vec{n}$ form an orthonormal basis of R^3. Then if $q' = (0, \vec{n}')$ and $q'' = (0, \vec{n}'')$, one has $(q' \cdot p) = (q'' \cdot p) = (q' \cdot q'') = 0$, and $\not{q}'^2 = q'^2 = \pm 1 = \not{q}''^2$. Therefore, the eigenvalues of $\not{q}'$ and $\not{q}''$ are in the set $\{i, -i\}$ or $\{1, -1\}$ depending on the sign convention for the metric.

One has $u_1 = \not{p}(u_0)$, then $\not{q}'u_1 = \not{p}(-\not{q}'u_0)$, but if $dim\ Im(\not{p}) = 1$, we must have $\not{q}'u_1 = \lambda' u_1$ with $\lambda' = \pm i$ (or ± 1), and similarly for $\not{q}''u_1$. Now $\not{q}'(\not{q}''u_1) = \lambda'\lambda'' u_1 = -\not{q}''(\not{q}'u_1) = -\lambda'\lambda'' u_1$. This would imply $\lambda'\lambda'' u_1 = 0$, which is impossible. Thus the only possibility is $dim\ Im(\not{p}) = 2 = dim\ Ker(\not{p})$, which implies $Im(\not{p}) = Ker(\not{p})$, and the theorem is proved also for $m = 0$. □

Theorem 2: *Let M be a $n \times n$ complex matrix. then one has:*

$$rank(M) = 1 \Leftrightarrow M \neq 0 \text{ and } \forall Q \in \mathcal{M}_n(C) \quad MQM = tr(MQ)M \tag{5.104}$$

Lemma: *Let M be a $n \times n$ complex matrix, such that $tr(M) \neq 0$. Then :*

$$rank(M) = 1 \Leftrightarrow M^2 = tr(M)M \tag{5.105}$$

Proof:

If M is a rank 1 matrix, then one can write $M = xy^\dagger$ where x and y are two complex vectors. Since we have $tr(M) = y^\dagger \cdot x$, one has $M^2 = tr(M)M$.

Conversely, since $tr(M) \neq 0$ then $X \wedge (X - tr(M)) = 1$, and the condition $M(M - tr(M)) = 0$ implies that $C^n = Ker(M) \oplus Ker(M - tr(M)I)$ $(n = 4)$, thanks to the kernel decomposition theorem. Thus, using a basis compatible with this decomposition, M can be written in the form:

$$M = P^{-1} \begin{pmatrix} 0 & & 0 & \\ & tr(M) & & \\ 0 & & \ddots & \\ & & & tr(M) \end{pmatrix} P \tag{5.106}$$

Where P is an invertible matrix. It leads to $tr(M) = dim\ Ker(M - tr(M)I) \times tr(M)$, thus $dim\ Ker(M - tr(M)I) = 1 \Rightarrow rank(M) = 1$. □

Proof of the theorem:

If $rank(M) = 1$, then $\forall Q,\ rank(MQ) = 0\ or\ 1$, and applying the same reasoning as in the lemma, we easily conclude that $MQM = tr(MQ)M$.

Conversely, if $\forall Q,\ tr(MQ) = 0$ then $M = 0$. Therefore, in our case, we can find Q such that $tr(MQ) \neq 0$. And since $Q \mapsto tr(MQ)$ is a continuous function, and the group of invertible matrices is dense in $\mathcal{M}_n(C)$, we can choose Q invertible. Then, multiplying on the right side by the matrix Q we get $(MQ)^2 = tr(MQ)MQ$ and the lemma tells us then that MQ is a rank 1 matrix, and thus $M = (MQ)Q^{-1}$ is also a rank 1 matrix. □

Corollary: *Let M be a $n \times n$ complex matrix ($M \neq 0$). We have the following equivalence:*

$$rank(M) = 1 \Leftrightarrow \forall (Q, Q')\ \ tr(MQMQ') = tr(MQ)tr(MQ') \tag{5.107}$$

Proof:

If one knows that M is a rank 1 matrix, the conclusion is a direct consequence of theorem 2. Conversely, if we know that $\forall (Q, Q')\ \ tr(MQMQ') = tr(MQ)tr(MQ')$, then the linear form $Q' \mapsto tr(MQMQ') - tr(MQ)tr(MQ')$ vanishes, which implies that $MQM = tr(MQ)M$ for every matrix Q. Then, the theorem 2 tells us that M is a rank 1 matrix. □

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